ANNALES ACADEMIAE SCIENTIARUM FENNICAE

Series A

I. MATHEMATICA

396

ON THE FOURTH COEFFICIENT OF UNIVALENT FUNCTIONS WITH BOUNDED BOUNDARY ROTATION

BY

M. SCHIFFER and O. TAMMI

HELSINKI 1967
SUOMALAINEN TIEDEAKATEMIA

31

ANNALES ACADEMIAE SCIENTIARUM FENNICAE

Series A

I. MATHEMATICA

396

ON THE FOURTH COEFFICIENT OF UNIVALENT FUNCTIONS WITH BOUNDED BOUNDARY ROTATION

BY

M. SCHIFFER and O. TAMMI

HELSINKI 1967
SUOMALAINEN TIEDEAKATEMIA

Communicated 14 October 1966 by Olli Lehto and K. I. Virtanen.

KESKUSKIRJAPAINO
HELSINKI 1967

1. Introduction

We shall consider in this paper regular functions in the unit disk $|z| < 1$ which have the normalized power series development

$$f(z) = z + a_2 z^2 + \ldots + a_n z^n + \ldots \tag{1}$$

and whose boundary rotation is bounded by the number

$$k\pi \ . \tag{2}$$

It is well known that for $2 \leq k \leq 4$ the functions $f(z)$ are univalent [1]. We denote the class of functions in question by S_k.

In a preceding paper, we developed a variational method to solve extremum problems within the class S_k [2]. It was shown that an extremum function $f(z)$ with maximal real coefficient a_n must satisfy the differential equation

$$1 + \frac{zf''(z)}{f'(z)} = \frac{G_n(z)}{g_n(z)} \tag{3}$$

where

$$\begin{cases} G_n(z) = \sum_{\nu=1}^{n-1} \frac{\nu^2 a_\nu}{z^{n-\nu}} + \sum_{\nu=1}^{n-1} \nu^2 \bar{a}_\nu \, z^{n-\nu} + n(n-1) a_n \, , \\ g_n(z) = \sum_{\nu=1}^{n-1} \frac{\nu a_\nu}{z^{n-\nu}} - \sum_{\nu=1}^{n-1} \nu \bar{a}_\nu \, z^{n-\nu} \, . \end{cases} \tag{4}$$

From this condition we deduced that the extremum function maps the unit disk onto a polygon with N corner points with

$$2 \leq N \leq 2n - 2 \ . \tag{5}$$

On the other hand, the Poisson-Stieltjes representation for the functions of the class S_k reduces in the case of such a polygonal mapping to the form

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1}{2} \sum_{i=1}^{N} \frac{z_i + z}{z_i - z} \gamma_i \ . \tag{6}$$

This work was supported in part by Air Force Contract AF 49 (638) 1345 at Stanford University.

Here the points z_i $(i = 1, \ldots, N)$ are the pre-images of the corner points, and $\pi\gamma_i$ is the angle of change of direction between the two sides which join at that corner. It was shown in [2] (formula (71)) that in the extremal case we have

$$(7) \qquad \begin{cases} \sum_{i=1}^{N} \gamma_i = 2 . \\ \sum_{i=1}^{N} |\gamma_i| = k . \end{cases}$$

Let us combine the representations (3) and (6) for the extremum function and write

$$(8) \qquad 1 + \frac{zf''(z)}{f'(z)} = \frac{G_n(z)}{g_n(z)} = \tfrac{1}{2}\sum_{\nu=1}^{2n-2} \frac{z_\nu + z}{z_\nu - z}\gamma_\nu .$$

Here the z_ν are now the roots of the equation

$$(9) \qquad g_n(z) = 0 .$$

The pre-images z_i are among the roots. If $N < 2n - 2$, the equation (9) has also roots which do not correspond to actual corner points of the polygon. We call these roots free roots. In order to satisfy the identity (8), we must define

$$(10) \qquad \gamma_\nu = 0 \text{ if } z_\nu \text{ is a free root.}$$

Observe that with this definition we can replace (7) by

$$(11) \qquad \sum_{\nu=1}^{2n-2} \gamma_\nu = 2 , \sum_{\nu=1}^{2n-2} |\gamma_\nu| = k .$$

From the identity (8) we can derive relations between the roots of equation (9) and the coefficients of the extremum function. Indeed, using the definitions (4), we find

$$(12) \qquad \frac{G_n(z) - zg_n'(z) - ng_n(z)}{g_n(z)} = \frac{n(n-1)a_n + 2n\sum_{\nu=1}^{n-1} \nu\bar{a}_\nu z^{n-\nu}}{\sum_{\nu=1}^{n-1} \frac{\nu a_\nu}{z^{n-\nu}} - \sum_{\nu=1}^{n-1} \nu\bar{a}_\nu z^{n-\nu}}$$

$$= \frac{n(n-1)a_n z^{n-1} + 2n\sum_{\nu=1}^{n-1} \bar{a}_\nu z^{2n-1-\nu}}{1 + 2a_2 z + \ldots} = n(n-1)a_n z^{n-1} + (z^n) .$$

On the other hand, from the definition (4) of $g_n(z)$ we can express the rational function in terms of its roots and write

$$g_n(z) = -\frac{1}{z^{n-1}} \prod_{\nu=1}^{2n-2} (z - z_\nu) . \tag{13}$$

Hence

$$z\frac{g_n'(z)}{g_n(z)} = -(n-1) + \sum_{\nu=1}^{2n-2} \frac{z}{z - z_\nu} = \sum_{\nu=1}^{2n-2} \frac{z + z_\nu}{2(z - z_\nu)}$$

and inserting this identity into (12) we find by virtue of (8):

$$\begin{aligned} \frac{G_n(z)}{g_n(z)} - z\frac{g_n'(z)}{g_n(z)} - n &= \tfrac{1}{2} \sum_{\nu=1}^{2n-2} (1 + \gamma_\nu) \frac{z_\nu + z}{z_\nu - z} - n \\ &= \sum_{\nu=1}^{2n-2} (1 + \gamma_\nu) \frac{z}{z_\nu - z} = \sum_{\nu=1}^{2n-2} (1 + \gamma_\nu) \sum_{\mu=1}^{\infty} \left(\frac{z}{z_\nu}\right)^\mu . \end{aligned} \tag{14}$$

Comparing the expansions (12) and (14) we find the following equations:

$$\begin{cases} \displaystyle\sum_{\nu=1}^{2n-2} \frac{1 + \gamma_\nu}{z_\nu^\mu} = 0 & (\mu = 1, 2, \ldots, n-2) , \\ \displaystyle\sum_{\nu=1}^{2n-2} \frac{1 + \gamma_\nu}{z_\nu^{n-1}} = n(n-1)\, a_n . \end{cases} \tag{15}$$

Moreover, we read off from equations (13) and (15) that

$$\prod_{\nu=1}^{2n-2} z_\nu = -1 \tag{16}$$

(cf. formulas (142) and (143) of [2]).

The numbers $1 + \gamma_\nu$ can be estimated by aid of (11). We denote the positive and negative jumps γ_ν by Δ_ν^+ and $-\Delta_\nu^-$, respectively (Δ_ν^+, $\Delta_\nu^- \geqq 0$). For them (11) yields

$$\Delta_\nu^+ \leqq \sum \Delta_\nu^+ = \frac{k}{2} + 1 , \Delta_\nu^- \leqq \sum \Delta_\nu^- = \frac{k}{2} - 1 ;$$

$$1 + \Delta_\nu^+ \leqq 2 + \frac{k}{2} , 1 - \Delta_\nu^- \geqq 2 - \frac{k}{2} .$$

Hence

$$0 \leqq 2 - \frac{k}{2} \leqq 1 + \gamma_\nu \leqq 2 + \frac{k}{2} . \tag{17}$$

If $N = 2n - 2$ or $N = 2n - 3$, all the roots z_ν lie on the unit circumference ($|z_\nu| = 1$) and we conclude from (15), (17) and (11):

$$|a_n| \leqq \frac{1}{n(n-1)} \sum_{\nu=1}^{2n-2} (1 + \gamma_\nu) = \frac{2}{n-1} < 1 \text{ for } n \geqq 4 . \tag{18}$$

Since we know that the maximum value of $|a_n|$ in the class S_k is larger than 1 for $k > 2$, we see that the cases $N = 2n - 2$ and $N = 2n - 3$ are excluded. In the case $n = 4$, which will now be discussed in detail, we may thus assume that $N \leqq 4$.

2. Direct estimation

From now on we shall deal with the problem of maximizing the fourth coefficient a_4 which may be assumed positive whithout any loss of generality. We may then restrict ourselves to discuss polygons with $N = 2$, 3 and 4 corners.

Let us denote the roots of $g_4(z) = 0$ by

$$z_1\,,\ z_2\,,\ z_3\,,\ z_4\,,\ z_5\,,\ z_6\,.$$

In the case $N = 4$ we are interested only in the case where

$$|z_5| \neq 1\,,\quad |z_6| \neq 1\,.$$

The structure of $g_4(z)$ implies that with any z also $\frac{1}{\bar{z}}$ is a root of $g_4(z) = 0$. Hence, we may denote

$$z_5 = re^{i\varphi}\,,\ z_6 = \frac{1}{\bar{z}_5} = \frac{1}{r}\,e^{i\varphi}\,. \tag{19}$$

In the case $N = 3$ we may assume z_5 and z_6 to have the same form (19). If z_4 is a free root of the equation $g_4(z) = 0$, it still must lie on the unit circumference as can be seen from equation (16).

Hence, in both cases $N = 3$ and $N = 4$ we may suppose the form (19) for the last two roots and write

$$z_1 = e^{i\varphi_1},\ z_2 = e^{i\varphi_2},\quad z_3 = e^{i\varphi_3},\ z_4 = e^{i\varphi_4}\,. \tag{20}$$

However, for $N = 4$ we have

$$\gamma_1\,,\ \gamma_2\,,\ \gamma_3\,,\ \gamma_4 \neq 0\,;\ \gamma_5 = \gamma_6 = 0\,;\ N = 4\,, \tag{21}$$

and for $N = 3$

$$\gamma_1\,,\ \gamma_2\,,\ \gamma_3 \neq 0\,;\ \gamma_4 = \gamma_5 = \gamma_6 = 0\,;\ N = 3\,. \tag{22}$$

We introduce the abreviation

$$\delta_\nu = 1 + \gamma_\nu \geqq 0 \quad (\nu = 1\,, 2\,, \ldots\,, 6) \tag{23}$$

(cf. (17)) and write the equations (15) in the case $n = 4$ in the form

$$\frac{\delta_1}{z_1} + \frac{\delta_2}{z_2} + \frac{\delta_3}{z_3} + \frac{\delta_4}{z_4} = -\left(r + \frac{1}{r}\right)e^{-i\varphi}, \tag{24}$$

$$\frac{\delta_1}{z_1^2} + \frac{\delta_2}{z_2^2} + \frac{\delta_3}{z_3^2} + \frac{\delta_4}{z_4^2} = -\left(r^2 + \frac{1}{r^2}\right)e^{-i2\varphi}, \tag{25}$$

$$12a_4 = \frac{\delta_1}{z_1^3} + \frac{\delta_2}{z_2^3} + \frac{\delta_3}{z_3^3} + \frac{\delta_4}{z_4^3} + \left(r^3 + \frac{1}{r^3}\right)e^{-i3\varphi}. \tag{26}$$

We deduce from (25) and (11) the estimate

$$r^2 + \frac{1}{r^2} \leqq \sum_{\nu=1}^{4} \delta_\nu = 4 + \sum_{\nu=1}^{4} \gamma_\nu = 6. \tag{27}$$

Hence, we infer

$$\left(r + \frac{1}{r}\right)^2 = r^2 + \frac{1}{r^2} + 2 \leqq 8. \tag{28}$$

This leads to the estimate

$$r^3 + \frac{1}{r^3} = \left(r + \frac{1}{r}\right)\left(r^2 + \frac{1}{r^2} - 1\right) \leqq 10\sqrt{2}. \tag{29}$$

Thus, finally, we find the following inequality for a_4 by using (26):

$$12a_4 \leqq \sum_{\nu=1}^{4} \delta_\nu + 10\sqrt{2} = 6 + 10\sqrt{2};$$

that is,

$$a_4 \leqq \tfrac{1}{2} + \tfrac{5}{6}\sqrt{2} < 1{,}6786. \tag{30}$$

On the other hand, consider the case $N = 2$. The extremal function is, in this case, easily determined (see Section 8). One finds for its fourth coefficient the expression

$$a_4 = \frac{k^3 + 8k}{24}. \tag{31}$$

This value for a_4 exceeds the upper bound (30) as long as

$$2,67 \leqq k \leqq 4. \tag{32}$$

Thus, as long as the boundary rotation satisfies the inequality (32), we are sure that the maximum for a_4 is achieved in the case $N = 2$ and that $|a_4|$ has the upper bound (31) for all functions of the class S_k.

Our aim is to prove that $N = 2$ yields the extremum function even for $2 \leqq k \leqq 4$.

? A sharper estimate

Multiplying both sides of (24), (25) and (26) by $e^{i\varphi}$, $e^{i2\varphi}$ and $e^{i3\varphi}$, respectively, we obtain

$$r + \frac{1}{r} = - \sum_{\nu=1}^{4} \delta_\nu \zeta_\nu \,, \tag{33}$$

$$r^2 + \frac{1}{r^2} = - \sum_{\nu=1}^{4} \delta_\nu \zeta_\nu^2 \,, \tag{34}$$

$$12 a_4\, e^{i3\varphi} = \sum_{\nu=1}^{4} \delta_\nu \zeta_\nu^3 + r^3 + \frac{1}{r^3} \tag{35}$$

with

$$\zeta_\nu = \frac{e^{i\varphi}}{z_\nu} = e^{ix_\nu} \qquad (\nu = 1\,, 2\,, 3\,, 4)\,. \tag{36}$$

From

$$\left\{\begin{aligned} r + \frac{1}{r} &= - \sum_{\nu=1}^{4} \delta_\nu \cos x_\nu \,, \\ r^2 + \frac{1}{r^2} &= - \sum_{\nu=1}^{4} \delta_\nu \cos 2x_\nu \,, \\ 6 &= \sum_{\nu=1}^{4} \delta_\nu \,, \end{aligned}\right. \tag{37}$$

we deduce

$$6 - \left(r^2 + \frac{1}{r^2}\right) = \sum_{\nu=1}^{4} \delta_\nu (1 + \cos 2x_\nu) = 2 \sum_{\nu=1}^{4} \delta_\nu \cos^2 x_\nu \,. \tag{38}$$

We apply the Schwarz inequality

$$\left(\sum_{\nu=1}^{4} \delta_\nu \cos x_\nu\right)^2 \leqq \sum_{\nu=1}^{4} \delta_\nu \sum_{\nu=1}^{4} \delta_\nu \cos^2 x_\nu = 6 \sum_{\nu=1}^{4} \delta_\nu \cos^2 x_\nu \tag{39}$$

and see that

$$\left(r + \frac{1}{r}\right)^2 \leqq 3 \left[6 - \left(r^2 + \frac{1}{r^2}\right)\right] .$$

Rearranging, we find

$$r^2 + \frac{1}{r^2} \leqq 4 \tag{40}$$

and since

$$\left(r + \frac{1}{r}\right)^2 = r^2 + \frac{1}{r^2} + 2 \leqq 6$$

we get

$$r + \frac{1}{r} \leqq \sqrt{6}\,. \tag{41}$$

This leads to

$$r^3 + \frac{1}{r^3} = \left(r + \frac{1}{r}\right)\left(r^2 + \frac{1}{r^2} - 1\right) \leqq 3\sqrt{6} \tag{42}$$

and thus we derive from (35) the estimate

$$a_4 \leqq \frac{6 + 3\sqrt{6}}{12} = \tfrac{1}{2} + \tfrac{1}{4}\sqrt{6} < 1,\ 11238\,. \tag{43}$$

This bound lies under the value (31) as long as

$$2{,}129692 \leqq k \leqq 4\,. \tag{44}$$

Thus the estimate (31) for a_4 is valid in this larger interval.

4. Discussion of the inequalities

It is clear that the bounds (40) and (41) *cannot be improved*. Indeed, the inequality sign entered in these estimates only through the use of the Schwarz inequality, and this inequality is sharp if we choose all four values of $\cos x_\nu$ to be equal. When selecting

$$\cos x_\nu = -\frac{1}{\sqrt{6}} \qquad (\nu = 1, \ldots, 4) \tag{45}$$

we see that (37) leads to the above upper bounds.

It should be observed, however, that our estimates were all based on the three conditions (33), (34) and (35) and that until now we have not taken into account the additional requirement (16).

We wish to show that the cases $N = 3$ and $N = 4$ can never lead to the actual extremum function. To show this, we have only to show that the necessary extremum conditions can not be satisfied if

$$r^3 + \frac{1}{r^3} > 6 \quad \text{and} \quad a_4 > 1\,, \tag{46}$$

since we know that the a_4 given by (31) has always larger value than 1.

We set up

$$\cos x_\nu = -\frac{1}{\sqrt{6}} + \varepsilon_\nu \quad (\nu = 1, 2, 3, 4) \tag{47}$$

and will show that the ε_ν must be small if (46) shall hold. In this case, we will show that condition (16) will be violated. Thus we will exclude the possibility of $N = 3$ or $N = 4$ for all k and show that (31) is the precise bound for $|a_4|$.

We denote

$$s = r + \frac{1}{r} \geqq 2 . \tag{48}$$

By (33) we have

$$s = -\sum_{\nu=1}^{4} \delta_\nu \cos x_\nu = \sqrt{6} - \sum_{\nu=1}^{4} \delta_\nu \varepsilon_\nu . \tag{49}$$

The central role in our estimation will be played by the quantity

$$c = \sum_{\nu=1}^{4} \delta_\nu \varepsilon_\nu . \tag{50}$$

In view of (49) we have

$$c = \sqrt{6} - s \leqq \sqrt{6} - 2 . \tag{51}$$

From (34) we can then deduce the chain of equalities

$$\begin{aligned} 2 - s^2 = \sum_{\nu=1}^{4} \delta_\nu \cos 2x_\nu &= 2 \sum_{\nu=1}^{4} \delta_\nu \cos^2 x_\nu - \sum_{\nu=1}^{4} \delta_\nu \\ &= 2 \sum_{\nu=1}^{4} \delta_\nu \left(-\frac{1}{\sqrt{6}} + \varepsilon_\nu \right)^2 - 6 \\ &= 2 \left(1 + \sum_{\nu=1}^{4} \delta_\nu \varepsilon_\nu^2 - \frac{2}{\sqrt{6}} c \right) - 6 \end{aligned}$$

whence

$$s^2 = 6 + \frac{4}{\sqrt{6}} c - 2 \sum_{\nu=1}^{4} \delta_\nu \varepsilon_\nu^2 . \tag{52}$$

Inserting on the left the value of s given by (51), we obtain

$$6 - 2\sqrt{6}\, c + c^2 = 6 + \frac{4}{\sqrt{6}} c - 2 \sum_{\nu=1}^{4} \delta_\nu \varepsilon_\nu^2$$

and

$$\frac{16}{\sqrt{6}} c = c^2 + 2 \sum_{\nu=1}^{4} \delta_\nu \varepsilon_\nu^2 . \tag{53}$$

We see that c is non-negative and can complete the estimate (51) to

$$0 \leqq c \leqq \sqrt{6} - 2 . \tag{54}$$

Moreover, we can estimate

$$\sum_{\nu=1}^{4} \delta_\nu \varepsilon_\nu^2 \leqq \frac{8}{\sqrt{6}} c . \tag{55}$$

This shows that the smallness of c will imply the smallness of all deviation terms ε_ν .

As stated above, we can assume without loss of generality that

$$r^3 + \frac{1}{r^3} > 6 . \tag{56}$$

In view of (51) we have

$$r^3 + \frac{1}{r^3} = \left(r + \frac{1}{r}\right)^3 - 3\left(r + \frac{1}{r}\right) = s^3 - 3s \tag{57}$$
$$= 3\sqrt{6} - 15c + 3\sqrt{6}\, c^2 - c^3 .$$

Thus we obtain from (56) the inequality for c:

$$y(c) = 3\sqrt{6} - 6 - 15c + 3\sqrt{6}\, c^2 - c^3 > 0 . \tag{58}$$

We need only to consider those values of c which lie in the interval (54). There $y(c)$ is a monotonic function of c and calculation shows that $y(11^{-1}) > 0$, $y(0{,}09418835) < 0$. Thus (58) yields the limitation for c:

$$0 \leqq c < 0{,}09418835 . \tag{59}$$

To be sure, we start with an accuracy which finally appears to be unnecessarily great to obtain the final contradiction.

5. An improved estimate for c.

We will now improve the bounds for c by using the second part of assumption (46), that is, $a_4 > 1$. We start with (33) in the form

$$\operatorname{Re}\{12\, a_4 e^{i3\eta}\} = \sum_{\nu=1}^{4} \delta_\nu \cos 3x_\nu + r^3 + \frac{1}{r^3} . \tag{60}$$

The first right-hand term can be transformed as follows:

$$\begin{aligned}
\sum_{\nu=1}^{4} \delta_\nu \cos 3x_\nu &= \sum_{\nu=1}^{4} \delta_\nu [4 \cos^3 x_\nu - 3 \cos x_\nu] \\
&= -3 \sum_{\nu=1}^{4} \delta_\nu \cos x_\nu + 4 \sum_{\nu=1}^{4} \delta_\nu \cos^3 x_\nu \\
&= 3\sqrt{6} - 3c + 4 \sum_{\nu=1}^{4} \delta_\nu \left(-\frac{1}{\sqrt{6}} + \varepsilon_\nu\right)^3 \\
&= \frac{7}{3}\sqrt{6} - c - 2\sqrt{6} \sum_{\nu=1}^{4} \delta_\nu \varepsilon_\nu^2 + 4 \sum_{\nu=1}^{4} \delta_\nu \varepsilon_\nu^3 .
\end{aligned}$$

By use of (53) we can replace

$$2 \sum_{\nu=1}^{4} \delta_\nu \varepsilon_\nu^2 = \frac{16}{\sqrt{6}} c - c^2$$

and find

$$\sum_{\nu=1}^{4} \delta_\nu \cos 3x_\nu = \frac{7}{3}\sqrt{6} - 17c + \sqrt{6}\, c^2 + 4 \sum_{\nu=1}^{4} \delta_\nu \varepsilon_\nu^3 . \tag{61}$$

Inserting this equation and (57) into (60) we get

$$R = \operatorname{Re}\{12\, a_4\, e^{i3\eta}\} = \frac{16}{3}\sqrt{6} - 32c + 4\sqrt{6}\, c^2 - c^3 + 4 \sum_{\nu=1}^{4} \delta_\nu \varepsilon_\nu^3 . \tag{62}$$

In order to discuss next $\operatorname{Im}\{12\, a_4\, e^{i3\eta}\}$ we observe that by virtue of (33) and (34) we have

$$\sum_{\nu=1}^{4} \delta_\nu \sin x_\nu = \sum_{\nu=1}^{4} \delta_\nu \sin 2x_\nu = 0 . \tag{63}$$

Thus

$$\begin{aligned}
I = \operatorname{Im}\{12\, a_4\, e^{i3\eta}\} &= \sum_{\nu=1}^{4} \delta_\nu (4 \cos^2 x_\nu - 1) \sin x_\nu \\
&= 4 \sum_{\nu=1}^{4} \delta_\nu \cos^2 x_\nu \sin x_\nu = 4 \sum_{\nu=1}^{4} \delta_\nu \left(\cos x_\nu + \frac{1}{\sqrt{6}}\right)^2 \sin x_\nu \\
&= 4 \sum_{\nu=1}^{4} \delta_\nu \sin x_\nu \varepsilon_\nu^2 .
\end{aligned}$$

Hence by use of (55) we find

$$| I | \leqq 4 \sum_{\nu=1}^{4} \delta_\nu \varepsilon_\nu^2 \leqq \frac{32}{\sqrt{6}} c . \tag{64}$$

Now we know that we can assume in our investigation

$$| 12\, a_4 e^{i3\varphi} |^2 = R^2 + I^2 > 144 .$$

Hence we have a fortiori the estimate

$$R^2 + \left(\frac{32}{\sqrt{6}}\, c\right)^2 > 144 , \tag{65}$$

that is,

$$R \left[1 + \left(\frac{32}{\sqrt{6}}\, \frac{c}{R}\right)^2\right]^{\frac{1}{2}} > 12 . \tag{66}$$

According to (59) we have an upper bound for c, and therefore:

$$\frac{32}{\sqrt{6}}\, c < \frac{32}{10\sqrt{6}} < 1{,}230472 .$$

By (65) this implies the lower estimate for R:

$$R^2 > 144 - 1{,}230472^2 > 142{,}4859.$$

Thus (66) yields

$$R \left[1 + \left(\frac{32}{10\sqrt{6}}\right)^2 \frac{1}{142{,}4859}\right]^{\frac{1}{2}} > 12$$

and we end up with the lower bound

$$R > 11{,}93674. \tag{67}$$

Insert this estimate into (62) and find the inequality

$$\frac{16}{3}\sqrt{6} - 32\, c + 4\sqrt{6}\, c^2 - c^3 + 4 \sum_{\nu=1}^{4} \delta_\nu \varepsilon_\nu^3 > 11{,}93674. \tag{68}$$

There remains still the problem to replace the sum on the left side by terms depending only on c. By means of (55) we get

$$\sum_{\nu=1}^{4} \delta_\nu \varepsilon_\nu^3 \leqq \max | \varepsilon_\nu | \cdot \sum_{\nu=1}^{4} \delta_\nu \varepsilon_\nu^2 \leqq \max | \varepsilon_\nu | \cdot \frac{8}{\sqrt{6}}\, c . \tag{69}$$

Furthermore, we have

$$\delta_\nu \varepsilon_\nu^2 \leqq \sum_{i=1}^{4} \delta_i \varepsilon_i^2 \leqq \frac{8}{\sqrt{6}}\, c \qquad (\nu = 1, 2, 3, 4)$$

whence

$$\max | \varepsilon_\nu | \leqq \sqrt{\frac{1}{\min \delta_\nu} \cdot \frac{8}{\sqrt{6}}\, c} . \tag{70}$$

To find $\min \delta_\nu$ we observe that we may restrict our attention to the case

$$2 \leqq k \leqq 2{,}129692\ . \tag{71}$$

In view of (17) we have hence

$$\min \delta_\nu \geqq 2 - \frac{2{,}129692}{2} = 0{,}935154. \tag{72}$$

Thus by use of (70) we obtain finally

$$\max |\varepsilon_\nu| \leqq \frac{1}{\sqrt{0{,}935154}} \left(\frac{8}{\sqrt{6}}\, c\right)^{\frac{1}{2}}. \tag{73}$$

We combine this with (69) to derive from (68) the inequality for c:

$$z(c) = \frac{16}{3}\sqrt{6} - 11{,}93674 - 32\,c \tag{74}$$

$$+ \frac{4}{\sqrt{0{,}935154}} \cdot \left(\frac{8}{\sqrt{6}}\, c\right)^{3/2} + 4\sqrt{6}\, c^2 - c^3 > 0.$$

It is easy to verify that $z(c)$ is a monotonically decreasing function of c in the interval (59). We compute that

$$y(24^{-1}) > 0 \quad \text{but} \quad y(0{,}04244641) < 0\ .$$

Thus we conclude that c cannot exceed the upper bound of the improved estimate

$$0 \leqq c < 0{,}04244641. \tag{75}$$

If we substitute this estimate into (73), we find

$$|\varepsilon_\nu| < 0{,}3850227. \tag{76}$$

6. The use of condition (16).

The necessary condition (16) which has not yet been used reads in the case $n = 4$ as follows

$$z_1 \cdot z_2 \cdot z_3 \cdot z_4 \cdot z_5 \cdot z_6 = -1\ . \tag{77}$$

With the rotation (19) and (36) this can be expressed in the form

$$e^{i(x_1 + x_2 + x_3 + x_4)} = -\, e^{6i\varphi} \tag{78}$$

or equivalently

$$\sum_{\nu=1}^{4} x_\nu = 6\varphi + (2n+1)\pi \qquad (n = 0, \pm 1, \ldots)\ . \tag{79}$$

We have to prove that this necessary condition cannot be fulfilled under the assumptions (46)

We begin with an estimate for φ. We use (64) and (75) to obtain

$$|I| \leqq \frac{32}{\sqrt{6}} c < \frac{32 \cdot 0{,}04244641}{2{,}129692} < 0{,}6377848 . \tag{80}$$

Since $I = 12 a_4 \sin 3\varphi$ this implies

$$\sin |3\varphi| < \frac{0{,}6377848}{12\, a_4} < \frac{0{,}6377848}{12} < 0{,}05314874 ;$$

here we used again our assumption $a_4 > 1$. We thus proved the inequalities

$$|3\varphi| < 0{,}0531738$$

and

$$|6\varphi| < 0{,}1063476 . \tag{81}$$

We introduce the angle x_0 by the conditions

$$\cos x_0 = -\frac{1}{\sqrt{6}} , \quad \frac{\pi}{2} < x_0 < \pi . \tag{82}$$

Since by (76)

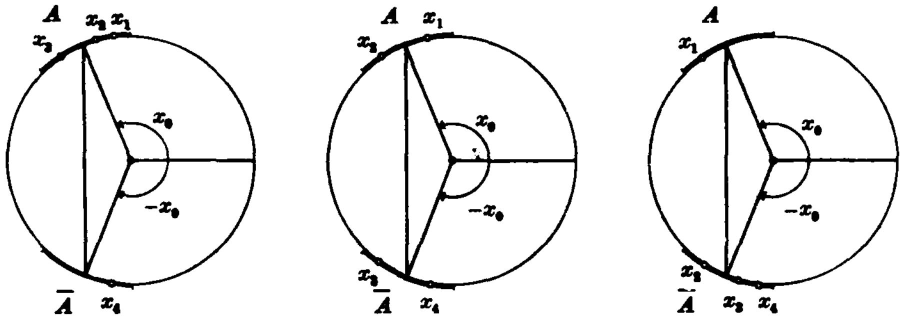
$$\cos x_\nu = \cos x_0 + \varepsilon_\nu , |\varepsilon_\nu| < 0{,}3850227 \quad (\nu = 1 , \ldots , 4) ,$$

we know that the points x_ν lie on the marked arcs A and $\bar{A}$ around the points x_0 and $-x_0$ in Figure 1. Since

$$\sum_{\nu=1}^{4} \delta_\nu \sin x_\nu = 0$$

it is impossible that all numbers $\sin x_\nu$ have the same sign. Thus in each arc A and $\bar{A}$ must lie at least one point x_ν. There are three possible cases to be distinguished as is shown in Figure 1. We denote the angular deviation of x_ν from the points x_0 or $-x_0$ in the arc A and $\bar{A}$, respectively, by η_ν. Thus we have the cases:

$$1^\circ \left| \begin{aligned} x_1 &= x_0 + \eta_1 \\ x_2 &= x_0 + \eta_2 \\ x_3 &= x_0 + \eta_3 \\ x_4 &= -x_0 + \eta_4 \end{aligned} \right. \qquad 2^\circ \left| \begin{aligned} x_1 &= x_0 + \eta_1 \\ x_2 &= x_0 + \eta_2 \\ x_3 &= -x_0 + \eta_3 \\ x_4 &= -x_0 + \eta_4 \end{aligned} \right. \qquad 3^\circ \left| \begin{aligned} x_1 &= x_0 + \eta_1 \\ x_2 &= -x_0 + \eta_2 \\ x_3 &= -x_0 + \eta_3 \\ x_4 &= -x_0 + \eta_4 . \end{aligned} \right.$$



1° 2° 3°

Figure 1.

Condition (79) takes different forms in the three cases; we find

$$(84)\qquad \left\{\begin{array}{ll} 1^\circ & \sum_{\nu=1}^{4} \eta_\nu + 2x_0 - 6\varphi = (2n+1)\,\pi\,, \\ 2^\circ & \sum_{\nu=1}^{4} \eta_\nu - 6\varphi = (2n+1)\,\pi\,, \\ 3^\circ & \sum_{\nu=1}^{4} \eta_\nu - 2x_0 - 6\varphi = (2n+1)\,\pi\,. \end{array}\right.$$

From definition (82) we find

$$(85)\qquad \begin{array}{l} 1{,}991330 < \ x_0 < 1{,}991331\ ; \\ 3{,}982660 < 2x_0 < 3{,}982662\,. \end{array}$$

We proceed now to estimate $|\eta_\nu|$ by use of our knowledge of the ε_ν. Since

$$\varepsilon_\nu = \cos x_\nu - \cos x_0 = -\,2 \sin \frac{x_\nu + x_0}{2} \sin \frac{x_\nu - x_0}{2}$$

we find

$$(86)\qquad |\,\varepsilon_\nu\,| = 2 \left|\sin \frac{x_\nu + x_0}{2}\right| \left|\sin \frac{x_\nu - x_0}{2}\right|.$$

Observe that $x_\nu - x_0 = \eta_\nu$ if x_ν lies in A and $x_\nu + x_0 = \eta_\nu$ if x_ν lies in $\bar{A}$. Denote

$$\cos \alpha = -\frac{1}{\sqrt{6}} - 0{,}3850227$$

and find

$$(87)\qquad \alpha < 2{,}4869590.$$

Thus

$$\sin\left|\frac{x_\nu \pm x_0}{2}\right| \geqq \sin\frac{x_0 + \alpha}{2} > \sin\frac{1{,}991331 + 2{,}4869590}{2} > 0{,}7848460\,.$$

By use of this estimate we deduce from (86) the inequality

$$\left|\sin\frac{\eta_\nu}{2}\right| < \frac{|\varepsilon_\nu|}{2\cdot 0{,}784846} < 0{,}2452856 \tag{88}$$

and arrive at the numerical bound

$$\left|\frac{\eta_\nu}{2}\right| < 0{,}2478143.$$

Since $\dfrac{\sin x}{x}$ decreases with increasing positive small x, we can assert

$$\frac{\left|\sin\frac{\eta_\nu}{2}\right|}{\left|\frac{\eta_\nu}{2}\right|} > \frac{\sin 0{,}2478143}{0{,}2478143} > \frac{0{,}2452855}{0{,}2478143}\,.$$

Hence, using the estimate (88) again we arrive at

$$\left|\frac{\eta_\nu}{2}\right| < \frac{0{,}2478143}{0{,}2452855}\left|\sin\frac{\eta_\nu}{2}\right| < \frac{0{,}2478143}{0{,}2452855}\cdot\frac{|\varepsilon_\nu|}{1{,}569692}\,;$$

$$|\eta_\nu| < K\,|\varepsilon_\nu| \quad \text{with} \quad K = 1{,}287272. \tag{89}$$

Now we use the Schwarz inequality to estimate $\Sigma\,\eta_\nu$, which occurs in (84). We find by use of (72), (55) and (75):

$$\left(\sum_{\nu=1}^{4}|\eta_\nu|\right)^2 \leqq 4\sum_{\nu=1}^{4}\eta_\nu^2 \leqq 4K^2\sum_{\nu=1}^{4}\varepsilon_\nu^2 \leqq \frac{4K^2}{\min\delta_\nu}\sum_{\nu=1}^{4}\delta_\nu\varepsilon_\nu^2$$

$$\leqq \frac{4K^2}{\min\delta_\nu}\cdot\frac{8}{\sqrt{6}}\,c < \frac{4\cdot 1{,}287272^2\cdot 8\cdot 0{,}04244641}{0{,}9351539\cdot\sqrt{6}} < 0{,}9825913.$$

Thus, ultimately we obtain

$$\sum_{\nu=1}^{4}|\eta_\nu| < 0{,}9912574. \tag{90}$$

For later use we can compute analogously

$$\left(\sum_{\nu=1}^{2}|\eta_\nu|\right)^2 \leqq 2\sum_{\nu=1}^{2}\eta_\nu^2 \leqq 2K^2\sum_{\nu=1}^{4}\varepsilon_\nu^2\,.$$

Thus, by the same chain of inequalities, we find

$$\sum_{\nu=1}^{2} |\eta_\nu| < \frac{1}{\sqrt{2}}\ 0{,}9912574 < 0{,}7009249. \tag{91}$$

From (81) and (90) we see that in the case $2°$

$$|2n+1|\pi \text{ should be } \leqq \sum_{\nu=1}^{4} |\eta_\nu| + |6\varphi| < 1{,}098 \quad (n = 0, \pm 1, \pm 2, \ldots).$$

The equation (84) is thus impossible in the case $2°$ and we are reduced to the cases $1°$ and $3°$. Since the signs of the η_ν and φ are free, there is no real difference between these two cases. We would have finished our argument if we could show that equation (84) is impossible also in the case $1°$.

By (81), (85) and (90) we see that in the case $1°$

$$1{,}098 > \sum_{\nu=1}^{4} |\eta_\nu| + |6\varphi| \geqq |(2n+1)\pi - 2x_0|$$

$$\geqq |2n+1|\pi - 2x_0 > |2n+1|\pi - 3{,}983.$$

This is impossible for $n = 1, 2, \ldots$. Similarly (84) in the case $1°$ gives for $n = -1, -2, \ldots$

$$1{,}098 > |2n+1|\pi + 2x_0 > |2n+1|\pi + 3{,}982$$

which again is impossible. Hence, the only alternative left is the equation (84) in the case $1°$ with $n = 0$.

7. The final argument.

We have now to consider only one possible extremum case with $N > 2$. In this case, the points x_1, x_2, x_3 lie on the upper arc A while x_4 lies on the lower arc $\bar{A}$ of Figure 1. We are now able to obtain more specific estimates for the numbers δ_4 and $|\eta_4|$ which are connected with the distinguished point x_4.

From the equation (cf. (33))

$$\sum_{\nu=1}^{4} \delta_\nu \sin x_\nu = 0$$

we deduce

$$\delta_4 |\sin x_4| = \delta_1 \sin x_1 + \delta_2 \sin x_2 + \delta_3 \sin x_3.$$

Here (cf. (87))

$$\sin x_i > \sin \alpha > \sin 2{,}4870 > 0{,}6088 \qquad (i = 1, 2, 3)$$

and hence

$$\delta_4 > 0{,}6088\,(\delta_1 + \delta_3 + \delta_3) = 0{,}6088\,(6 - \delta_4)\,.$$

We find, therefore, the lower bound for δ_4:

$$\delta_4 > \frac{6 \cdot 0{,}6088}{1 + 0{,}6088} > 2{,}2705\,. \tag{92}$$

Next, we shall improve our information regarding ε_4. We start with the equations (cf. (33), (34) and (47))

$$\begin{cases} \sum_{\nu=1}^{4} \delta_\nu \sin x_\nu = \sum_{\nu=1}^{4} \delta_\nu \sin x_\nu \cos x_\nu = 0\,, \\ \cos x_\nu = \cos x_0 + \varepsilon_\nu\,. \end{cases}$$

They imply

$$\sum_{\nu=1}^{4} \delta_\nu \varepsilon_\nu \sin x_\nu = 0\,. \tag{93}$$

From the definition (50) of c there follows

$$c \sin x_0 = \sum_{\nu=1}^{4} \delta_\nu \varepsilon_\nu \sin x_0$$

and subtracting the equation (93) from this equation, we find

$$c \sin x_0 = \sum_{\nu=1}^{4} \delta_\nu \varepsilon_\nu\, 2 \sin \frac{x_0 - x_\nu}{2} \cos \frac{x_0 + x_\nu}{2}\,. \tag{94}$$

Since

$$\varepsilon_\nu = \cos x_\nu - \cos x_0 = 2 \sin \frac{x_0 - x_\nu}{2} \sin \frac{x_0 + x_\nu}{2}$$

we may bring (94) into the form

$$c \sin x_0 = \sum_{i=1}^{3} \delta_i \varepsilon_i^2 \cot \frac{x_0 + x_i}{2} + \delta_4 \varepsilon_4 (\sin x_0 - \sin x_4)$$

whence

$$\delta_4 \varepsilon_4 (\sin x_0 - \sin x_4) = - \sum_{i=1}^{3} \delta_i \varepsilon_i^2 \cot \frac{x_0 + x_i}{2} + c \sin x_0\,. \tag{95}$$

Now, since we are in the case 1°, we have

$$\sin x_0 > 0\,,\ \sin x_0 - \sin x_4 > 0\,,\ \cot \frac{x_0 + x_i}{2} < 0 \ (i = 1\,, 2\,, 3)\,.$$

Thus we read off from (95) the important information

(96) $$\varepsilon_4 > 0\,, \quad \eta_4 > 0\,.$$

We can utilize (95) also to find an upper bound for ε_4. We get

(97) $$\delta_4\varepsilon_4 (\sin x_0 + |\sin x_4|) = \sum_{i=1}^{3} \delta_i \varepsilon_i^2 \left|\cot \frac{x_0 + x_i}{2}\right| + c \sin x_0\,.$$

Clearly,

$$\sin x_0 + |\sin x_4| > \sin x_0 + \sin \alpha > 0{,}9125 + 0{,}6088 = 1{,}5213.$$

From (92) follows then

$$\delta_4 (\sin x_0 + |\sin x_4|) > 2{,}2705 \cdot 1{,}5213 > 3{,}4541$$

and (97) leads to

(98) $$3{,}4541\ \varepsilon_4 < \sum_{i=1}^{3} \delta_i \varepsilon_i^2 \left|\cot \frac{x_0 + x_i}{2}\right| + c \sin x_0\,.$$

Next, observe that

$$\left|\cot \frac{x_0 + x_i}{2}\right| < \left|\cot \frac{x_0 + \alpha}{2}\right| < \left|\cot \frac{1{,}9920 + 2{,}4870}{2}\right|$$
$$< |\cot 2{,}2395| < 0{,}7902$$

and that

$$\sum_{i=1}^{3} \delta_i \varepsilon_i^2 \leqq \sum_{\nu=1}^{4} \delta_\nu \varepsilon_\nu^2 \leqq \frac{8}{\sqrt{6}}\, c\,.$$

Hence (98) yields

$$3{,}4541\ \varepsilon_4 < \left(\frac{8}{\sqrt{6}} \cdot 0{,}7902 + \sin x_0\right) c$$
$$< (8 \cdot 0{,}7902 \cdot 0{,}4083 + 0{,}9130)\, 0{,}0425 < 0{,}1486\,.$$

Thus we proved

(99) $$0 < \varepsilon_4 < \frac{0{,}1486}{3{,}4541} < 0{,}0431\,.$$

By inequality (89) we can translate each estimate for ε_4 into a corresponding estimate for η_4. We find

(100) $$0 < \eta_4 < 1{,}2873 \cdot 0{,}0431 < 0{,}0555\,.$$

We are now in the position to dispose of the remaining condition (84) in the case 1° for $n = 0$. We write it in the form

(101) $$\sum_{\nu=1}^{4} \eta_\nu - 6\varphi = -2x_0 + \pi\,.$$

We distinguish the essentially different cases:

a) All η_i $(i = 1, 2, 3)$ have the same sign.
b) Two η_i are non-negative, one is negative.
c) One η_i is non-negative, two are negative.

We begin with case a). We assume that all η_i are non-positive; the case that they are all non-negative is treated in precisely the same manner. In view of (93) we have

$$\sum_{i=1}^{3} \delta_i \varepsilon_i \sin x_i = \delta_4 \varepsilon_4 \, | \sin x_4 | \, . \tag{102}$$

By (17) and (71) we have the upper bound for δ_4:

$$\delta_4 \leqq 2 + \frac{2{,}1297}{2} < 3{,}0649 \, .$$

Hence, by (99),

$$\sum_{i=1}^{3} \delta_i \varepsilon_i \sin x_i \leqq 3{,}0649 \cdot 0{,}0431 < 0{,}1321 \, . \tag{103}$$

On the other hand, by (72)

$$\delta_i \sin x_i > 0{,}9351 \sin x > 0{,}9351 \cdot 0{,}6088 > 0{,}5692 \, .$$

Thus (103) gives

$$0{,}5692 \sum_{i=1}^{3} \varepsilon_i < 0{,}1321 \, .$$

From (89) we then deduce

$$\sum_{i=1}^{3} | \eta_i | \leqq 1{,}2873 \sum_{i=1}^{3} \varepsilon_i < \frac{1{,}2873 \cdot 0{,}1321}{0{,}5692}$$

and obtain

$$\sum_{i=1}^{3} | \eta_i | < 0{,}2988 \, . \tag{104}$$

Combining (81), (100) and (104) we get

$$\sum_{\nu=1}^{4} | \eta_\nu | + | 6\varphi | < 0{,}4607 \, . \tag{105}$$

On the other hand,

$$| - 2x_0 + \pi | > 2x_0 - \pi > 3{,}9826 - 3{,}1416 = 0{,}8410 \tag{106}$$

and thus, according to (101), the number

$$0{,}4676 \text{ should be } > 0{,}8410.$$

Thus the case a) cannot occur.

We come now to case b): $\eta_1 < 0$, $\eta_2 \geqq 0$, $\eta_3 \geqq 0$. This implies $\varepsilon_1 > 0$, $\varepsilon_2 \leqq 0$, $\varepsilon_3 \leqq 0$. — We write (101) in the form

$$2x_0 - \pi + \sum_{\nu=1}^{4} \eta_\nu = 6\varphi . \tag{107}$$

From (89) and (76) we have the estimate valid for all η_ν:

$$| \eta_\nu | < 1{,}2873 \cdot 0{,}3851 < 0{,}4958. \tag{108}$$

Thus, since $2x_0 - \pi > 0{,}8410$, we know that the left side of (107) is positive. We have indeed the estimate

$$2x_0 - \pi + \sum_{\nu=1}^{4} \eta_\nu > 0{,}8410 - 0{,}4958 = 0{,}3452.$$

Because of (107) and (81) the number

$$0{,}1064 \text{ should be } > 0{,}3452.$$

This excludes the case b).

There remains then the case c): $\eta_1 \geqq 0$, $\eta_2 < 0$, $\eta_3 < 0$. This implies $\varepsilon_1 \leqq 0$, $\varepsilon_2 > 0$, $\varepsilon_3 > 0$. — Now we make use of (91) and find

$$| \eta_2 | + | \eta_3 | < 0{,}7010. \tag{109}$$

Thus the left side of (107) is still positive and satisfies the inequality

$$2x_0 - \pi + \sum_{\nu=1}^{4} \eta_\nu > 0{,}8410 - 0{,}7010 = 0{,}14.$$

In view of (107) and (81) the number

$$0{,}11 \text{ should be } > 0{,}14.$$

The last possibility has thus been excluded and we have shown that the number of corner points in the extremal polygon is precisely $N = 2$.

8. The case $N = 2$.

We have finally to discuss the case $N = 2$. Here we may derive from (7) the information (suppose that $\gamma_1 > 0$, $\gamma_2 < 0$):

$$\gamma_1 = \frac{k}{2} + 1 , \qquad \gamma_2 = - \left(\frac{k}{2} - 1 \right) . \tag{110}$$

Hence by the representation formula (6) we find easily

$$\frac{f''(z)}{f'(z)} = \frac{\gamma_1}{z_1 - z} + \frac{\gamma_2}{z_2 - z}\,, \tag{111}$$

where z_1 and z_2 are the pre-images of the corresponding corner points. Integrating (111) and using the normalization (1), we find

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} = \left(1 - \frac{z}{z_1}\right)^{-\gamma_1}\left(1 - \frac{z}{z_2}\right)^{-\gamma_2}. \tag{112}$$

If we develop the right expression into a power series and compare the coefficients of z^3 on both sides, we obtain

$$24\, a_4 = \gamma_1(\gamma_1 + 1)(\gamma_1 + 2) z_1^{-3} \tag{113}$$

$$+ 3\gamma_1\gamma_2[(\gamma_1 + 1) z_1^{-2} z_2^{-1} + (\gamma_2 + 1) z_1^{-1} z_2^{-2}] + \gamma_2(\gamma_2 + 1)(\gamma_2 + 2) z_2^{-3}.$$

Write this in the form

$$24\, a_4 z_1^3 = \gamma_1(\gamma_1 + 1)(\gamma_1 + 2) \tag{114}$$

$$+ 3\gamma_1\gamma_2[(\gamma_1 + 1)t + (\gamma_2 + 1)t^2] + \gamma_2(\gamma_2 + 1)(\gamma_2 + 2)t^3$$

where

$$t = \frac{z_1}{z_2}\,. \tag{115}$$

Because $z_1 = z_2$ leads to the extremal case for convex domains, for which $|a_n| = 1$, we see that the right side of (114) assumes the value 24 for $t = 1$. Thus we get

$$24\,(a_4 z_1^3 - 1) \tag{116}$$

$$= 3\gamma_1\gamma_2[(\gamma_1 + 1)(t - 1) + (\gamma_2 + 1)(t^2 - 1)] + \gamma_2(\gamma_2 + 1)(\gamma_2 + 2)(t^3 - 1)\,.$$

For brevity, write this in the form

$$24\,(a_4 z_1^3 - 1) = \gamma_2\, \Phi(t)\,, \tag{117}$$

where

$$\Phi(t) = (t - 1)\{3\gamma_1[\gamma_1 + 1 + (\gamma_2 + 1)(t + 1)] + (\gamma_2 + 1)(\gamma_2 + 2)(t^2 + t + 1)\}\,. \tag{118}$$

From (117) follows

$$24\,(|a_4| - 1) \leq |\gamma_2|\,|\Phi(t)|\,. \tag{119}$$

We want to maximize the right side of (119) and hope that the upper bound for $|a_4|$ thus obtained appears to be sharp.

For brevity, write

$$\Phi(t) = (t-1)\,(a + bt + ct^2)\ , \tag{120}$$

where

$$\begin{cases} a = 12\gamma_1 + (\gamma_2 + 1)\,(\gamma_2 + 2) = 18 + \dfrac{7}{2}k + \dfrac{k^2}{4}\,, \\ b = 2(\gamma_1 + 2)\,(\gamma_2 + 1) = \left(2 - \dfrac{k}{2}\right)(6 + k)\,, \\ c = (\gamma_2 + 1)\,(\gamma_2 + 2) = \left(2 - \dfrac{k}{2}\right)\left(3 - \dfrac{k}{2}\right). \end{cases} \tag{121}$$

Because $\overline{\Phi(t)} = \Phi(t^{-1})$, we obtain

$$\begin{aligned} |\,\Phi\,(t)\,|^2 &= \Phi(t)\cdot\Phi(t^{-1}) \\ &= (2 - t - t^{-1})\,[a^2 + b^2 + c^2 + b(a + c)\,(t + t^{-1}) + ca(t^2 + t^{-2})]\,. \end{aligned}$$

Define the new real variable

$$u = 1 - \frac{1}{2}\,(t + t^{-1})\,; \quad 0 \leqq u \leqq 2\,. \tag{122}$$

In u we get

$$|\,\Phi(t)\,|^2 = 2u\,[(a + b + c)^2 - 2(ab + bc + 4ca)u + 4cau^2]\ .$$

Thus we have to maximize the function

$$\frac{|\,\Phi(t)\,|^2}{2} = \psi(u) = Au + Bu^2 + Cu^3\,, \tag{123}$$

with

$$\begin{cases} A = (a + b + c)^2 = 36^2\,, \\ B = -\,2(ab + bc + 4ca) = -\,36(4 - k)\,(10 + k)\,, \\ C = 4ca = (4 - k)\,(6 - k)\left(18 + \dfrac{7}{2}k + \dfrac{k^2}{4}\right). \end{cases} \tag{124}$$

We have to show that $\psi(u)$ takes its maximum at $u = 2$ for all k-values interesting us: $2 < k < 4$. We compute:

$$\psi'(u) = A + 2Bu + 3Cu^2\,.$$

$\psi'(0) = A > 0$. Let us show that the discriminant of the equation $\psi'(u) = 0$ is negative; then $\psi'(u)$ cannot change its sign and our proof is finished. Thus we must have

(125) $$B^2 - 3AC < 0 \ ;$$

$$36^2(4-k)^2\,(10+k)^2 < 3 \cdot 36^2\,(4-k)\,(6-k)\left(18+\frac{7}{2}\,k+\frac{k^2}{4}\right).$$

This leads to

(126) $$4(4-k)\,(10+k)^2 < 3(6-k)\,(72+14k+k^2)\,.$$

Let us here replace $72 + 14k + k^2$ by its minimum value obtained for $k = 2$, i.e., by 104. Thus we are lead to the more restrictive condition

(127) $$(4-k)\,(10+k)^2 \leqq 72(6-k)\,.$$

Rearrange and find

$$k^3 + 16k^2 - 52k + 32 \geqq 0\,;$$

(128) $$(k-2)\,(k^2+18k-16) \geqq 0\,.$$

Hence there remains only to show that

(129) $$k^2 + 18\,k - 16 \geqq 0$$

for $2 < k < 4$. This quadratic expression is negative for $k = 0$, but already for $k = 2$ it has the value 24 and it remains positive for $k > 2$. Hence we proved (129) and since the order of our conclusions can be reversed from (129) to (125), we have shown that

(130) $$\max \varphi(u) = \varphi(2) = 2(A + 2B + 4C) = 2(12 + 2k + k^2)^2\,.$$

Hence

$$\max |\,\Phi(t)\,| = 24 + 4\,k + 2\,k^2$$

and from (119) we obtain the condition

$$|\,a_4\,| \leqq \frac{k^3+8k}{24}\,.$$

The equality is true only if $u = 2$, which means

$$t = \frac{z_1}{z_2} = -\,1\,.$$

Our result, which generalizes a theorem for the real class S_k[3], is thus:

Theorem. *In the class* $S_k\,(2 \leqq k \leqq 4)$ *of univalent functions with bounded boundary rotation, the coefficient* a_4 *satisfies the inequality*

$$|\,a_4\,| \leqq \frac{k^3+8k}{24}\,.$$

The extremum functions are

$$f(z) = \frac{1}{\tau k}\left[\left(\frac{1+\tau z}{1-\tau z}\right)^{k/2} - 1\right] \quad (|\tau| = 1).$$

References.

[1] PAATERO, V.: Über die konforme Abbildung von Gebieten deren Ränder von beschränkter Drehung sind. - Ann. Acad. Sci. Fenn. Ser. A 33 n:o 9 (1931).

[2] SCHIFFER, M. — TAMMI, O.: A method of variations for functions with bounded boundary rotation. - Journal d'Analyse Mathématique, vol. XVII, pp. 109–144 (1966).

[3] TAMMI, O.: Note on symmetric schlicht domains of bounded boundary rotation. - Ann. Acad. Sci. Fenn. Ser. A I n:o 198 (1955).

Stanford, California, U.S.A.
Helsinki, Finland.

Printed January 1967